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THE HYDRODYNAMIC WALL EFFECT FOR A DISPERSE SYSTEM

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Abstract—The flow of a highly dilute suspension of spheres (radius a) between two parallel ridid planes (distance L) in slow shearing motion is studied. Even for the limiting situation, (a/L) small but finite, there is a layer-one sphere diameter thick—immediately adjacent to the wall in which bulk quantities are so complicated functionals of the parameters of the microstructure that evaluating them seems out of the question. Nevertheless, it is still simple to obtain average bulk quantities (e.g. apparent viscosity) and even the evaluation of local bulk quantities far away from the wall poses no problem. The reason being that the customary continuum constitutive equation for the bulk stress can be utilized, although a slip velocity has to supplement it. This applies to any disperse system and can be applied to different flows, too. For the spherical suspension at hand an explicit expression for this slip velocity is obtained.

1. INTRODUCTION

Disperse systems, be it suspensions, emulsions, dispersions of macromolecular solutions, comprise a large group of materials of industrial importance. For many engineering fields it is of great importance to know how they respond to imposed forces or motions at their boundaries. This, however, is an extremely complicated problem even in situations where inertial effects play no role.

To see where the crux of the problem comes in we must realize that any disperse system may be regarded as a macro-continuum, provided the particle dimensions (say a) are small in comparison with the dimensions (say L) of the walls bounding the system. Assuming this to be the case one can, for a given microstructure, determine the rheological properties of this continuum (at least in principle), i.e. a relation between its bulk properties. This relation, called constitutive equation characterizes the macro-continuum. Close to a wall, however, the bulk properties are not only influenced by the microstructure of the disperse system but also by the wall. Admittedly, for $(a/L) \ll 1$ this wall layer, where conditions are untypical, will be thin. Nerertheless, since the $(a/L) \to 0$ limit, which the constitutive equation demands, can never be achieved in practice, it is important to know what happens if (a/L) is small, but finite.

To this end we study the flow of a dilute suspension of neutrally buoyant spheres (radius a) between two rigid planes (distance L) in steady relative shearing motion. We shall assume that all conditions, which the constitutive equation of Einstein demands, are met. The bulk viscosity μ of the suspension thus is $\mu_s(1 + 2.5\varphi_0)$, where φ_0 denotes the volume fraction ($\varphi_0 \ll 1$) and μ_s the viscosity of the solvent.

The most important quantity of the suspension will be the shear stress. Knowing that it has to be constant implies that one can use its asymptotic, i.e. far away (from the wall) value. It is thus given by the Einstein relation. The problem now consists in determining the far away shear rate, called $q^{(\infty)}$, which appears in that expression. This is not to be confused with the applied shear rate. As a matter of fact by extrapolating (the unknown) $q^{(\infty)}$ up to the wall, a relation to the known applied shear rate emerges. The difference between $q^{(\infty)}$ and the applied shear rate can be written as $(2u_s/L)$,† where for obvious reasons u_s is called slip velocity. Under the customary assumption that the true suspension adheres to the wall an explicit expression for u_s is derived. It is demonstrated that u_s does not require detailed knowledge of the bulk velocity close to the wall.

If V_w denotes the velocity of the moving plane the slip velocity reads

$$u_s = \lambda V_w \frac{a}{L} \varphi_0, \tag{1.1}$$

†Note that we have two walls.

with a dimensionless slip coefficient λ. This coefficient is estimated as 1.45. Being positive the suspension will thus show the so called sigma-phenomenon (e.g. Goldsmith & Mason 1967). Consequently, even though the spheres are uniformly distributed over the region accessible to them the apparent viscosity (ratio of bulk shear stress to applied shear rate) is lower in small instruments than in larger ones. The presence of a wall, modifying the behavior of a single sphere (and thus of the whole suspension) is responsible for that effect.

The hydrodynamic wall effect for a dilute suspension of spheres has also been studied by Guth & Simha (1936) and by Vand (1948). While Vand lists a slip coefficient of 3.25 the slip coefficient of Guth–Simha's result for 2-dim. shear (they considered an extensional flow) would be -1.05. It is demonstrated that in both articles the naive handwaving arguments employed and not computational errors led to erroneous results. The most comprehensive study of wall effects is due to Tözeren & Skalak (1977). They calculated bulk properties for arbitrary distances from a wall. Based on these minute calculations one can follow an alternate (but less efficient) route to obtain u_s, and extrapolate their numerical results for the far away bulk velocity up to the wall.

The concept of a slip velocity applies to any disperse system (e.g. Cox and Brenner, 1971). With the exception of evaluating local bulk quantities close to the boundary, it is needed to obtain the proper bulk quantities far away from the wall as well as for the evaluation of average global properties (like the apparent viscosity). More explicitely these quantities can indeed be evaluated by use of a customary constitutive equation (which is valid only for unbounded systems) provided a slip velocity u_s is introduced. This has nothing to do with slippage along the wall of the actual disperse systems. Quite to the contrary, the explicit expression for the slip velocity u_s derived in this paper rests on the adherence requirement of the suspension. Once u_s is known for 2-dim. shear the effects of a small but finite (a/L)-ratio for different types of flow are readily estimated.

2. THE HYDRODYNAMIC PROBLEM

Let us consider a single sphere (radius a, surface S_0, volume V_0) immersed in an incompressible Newtonian fluid (viscosity μ_s) if a plane wall W (unit normal $\mathbf{n}$ pointing towards the fluid) bounds this system. In general, the motion of the sphere will consist of a translational velocity of its center $\mathbf{r}_0$ (denoted by $\mathbf{u}_0$) and an angular velocity $\boldsymbol{\omega}$. Assuming inertial effects to be negligible the mathematical formulation of our problem reads

$$\frac{\partial}{\partial \mathbf{r}} \cdot \mathbf{u} = 0, \qquad [2.1a]$$

$$-\frac{\partial p}{\partial \mathbf{r}} + \mu_s \nabla^2 \mathbf{u} = 0, \qquad [2.1b]$$

subject to the boundary conditions

$$\mathbf{u} = \mathbf{u}_0 + \boldsymbol{\omega} \times (\mathbf{r} - \mathbf{r}_0) \quad \text{on} \quad S_0, \qquad [2.2]$$

$$\mathbf{u} = 0 \quad \text{on} \quad W, \qquad [2.3]$$

$$\left.\begin{matrix} \mathbf{u} \to \mathbf{u}^{(\infty)}(\mathbf{r}) \\ p \to p^{(\infty)}(\mathbf{r}) \end{matrix}\right\} |\mathbf{r} - \mathbf{r}_0| \to \infty, \mathbf{r} \notin W. \qquad [2.4]$$

Here $(\mathbf{u}^{(\infty)}, p^{(\infty)})$ stand for the velocity and pressure field, respectively, if the sphere is absent. These may be arbitrary except for the requirement that they satisfy the creeping motion equations [2.1] as well as the adherence condition [2.3] on the wall. This being the case we can introduce the disturbance fields $(\mathbf{v}, \bar{p})$,

$$\mathbf{v} = \mathbf{u} - \mathbf{u}^{(\infty)},$$

$$\bar{p} = p - p^{(\infty)}. \qquad [2.5]$$

They, too, satisfy the differential equations [2.1a, b] but the appropriate boundary conditions are

$$\mathbf{v} = \mathbf{u}_0 + \boldsymbol{\omega} \times (\mathbf{r} - \mathbf{r}_0) - \mathbf{u}^{(\infty)} \quad \text{on} \quad S_0, \qquad [2.6]$$

$$\mathbf{v} = 0 \quad \text{on} \quad W, \qquad [2.7]$$

$$\left.\begin{matrix} \mathbf{v} \to 0 \\ \bar{p} \to 0 \end{matrix}\right\} |\mathbf{r} - \mathbf{r}_0| \to \infty, \mathbf{r} \notin W. \qquad [2.8]$$

This corresponds to the problem of a sphere with a prescribed velocity at its surface S_0, suspended in a quiescent Newtonian fluid which is bounded by a stationary plane wall W.

Only the variation of $\mathbf{u}^{(\infty)}$ over the particle surface is needed. But $\mathbf{u}^{(\infty)}$ can vary substantially only over macroscopic dimensions, i.e. dimensions which are large in comparison to the sphere radius a. Consequently, as far as [2.6] is concerned a Taylor expansion around the sphere center seems appropriate and we put

$$\mathbf{u}^{(\infty)}(\mathbf{r}) = \mathbf{u}^0 + \boldsymbol{\Omega} \times (\mathbf{r} - \mathbf{r}_0) + (\mathbf{r} - \mathbf{r}_0) \cdot \mathbf{e}, \qquad [2.9]$$

where $\boldsymbol{\Omega}$ is one half the vorticity vector and $\mathbf{e}$ the rate of strain dyadic (all quantities evaluated at $\mathbf{r}_0$). Equation [2.6] now reads

$$\mathbf{v} = (\mathbf{u}_0 - \mathbf{u}^0) + (\boldsymbol{\omega} - \boldsymbol{\Omega}) \times (\mathbf{r} - \mathbf{r}_0) - (\mathbf{r} - \mathbf{r}_0) \cdot \mathbf{e} \quad \text{on} \quad S_0. \qquad [2.10]$$

The interpretation of this equation is obvious.

If $\mathbf{t}^f$ denotes the stress tensor of the fluid the force $\mathbf{F}$, couple $\mathbf{G}$ and stresslet $\mathbf{s}$ (the latter two with respect to $\mathbf{r}_0$) which are exerted upon the particle (surface normal $\mathbf{n}_s$) are defined as

$$\mathbf{F} = \int_{S_0} \mathrm{d}A \;\, \mathbf{n}_s \cdot \mathbf{t}^f, \qquad [2.11a]$$

$$\mathbf{G} = \int_{S_0} \mathrm{d}A \;\, (\mathbf{r} - \mathbf{r}_0) \times \mathbf{n}_s \cdot \mathbf{t}^f, \qquad [2.11b]$$

$$\mathbf{s} = \boldsymbol{\Delta}^{(2)} : \int_{S_0} \mathrm{d}A \;\, [\mathbf{n}_s \cdot \mathbf{t}^f(\mathbf{r} - \mathbf{r}_0) - 2\mu_s \mathbf{n}_s \mathbf{u}]. \qquad [2.11c]$$

Here $\boldsymbol{\Delta}^{(2)}$ is a fourth order tensor which upon operation on any second order tensor extracts the symmetric and irreducible part of that tensor. Its cartesian components are thus

$$\Delta^{(2)}_{ij,kl} = \frac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) - \frac{1}{3}\delta_{ij}\delta_{kl}.$$

For later reference we should point out an alternate form for the stresslet, namely

$$\mathbf{s} = \boldsymbol{\Delta}^{(2)}\!:\!\int_{V_0} \mathrm{d}V \mathbf{t}^{p}. \tag{2.12}$$

This simple relation is valid for any rigid particle, which is freely suspended ($\mathbf{F} = 0$, $\mathbf{G} = 0$). The integration of $\mathbf{t}^{(p)}$, i.e. the particle stress tensor, extends over the volume V_0 of the particle.

For our problem, the linearity of the differential equations [2.1] as well as of the boundary conditions [2.7], [2.8] and [2.10] implies that the result must be of the form†

$$\mathbf{F} = \mu_s\,\{{}^t\mathbf{K}\cdot(\mathbf{u}^0 - \mathbf{u}_0) + {}^t\mathbf{R}\cdot(\boldsymbol{\Omega} - \boldsymbol{\omega}) + {}^t\mathbf{Q}\!:\!\mathbf{e}\}, \tag{2.13a}$$

$$\mathbf{G} = \mu_s\,\{(\mathbf{u}^0 - \mathbf{u}_0)\cdot{}^t\mathbf{R} + {}^r\mathbf{R}\cdot(\boldsymbol{\Omega} - \boldsymbol{\omega}) + {}^r\mathbf{Q}\!:\!\mathbf{e}\}, \tag{2.13b}$$

$$\mathbf{s} = \mu_s\,\{(\mathbf{u}^0 - \mathbf{u}_0)\cdot{}^t\mathbf{Q} + (\boldsymbol{\Omega} - \boldsymbol{\omega})\cdot{}^r\mathbf{Q} + \mathbf{D}\!:\!\mathbf{e}\}, \tag{2.13c}$$

with

$${}^tK_{ij} = {}^tK_{ji},\; {}^rR_{ij} = {}^rR_{ji},\; D_{ijke} = D_{klij}. \tag{2.14}$$

Without the presence of a wall the tensors appearing on [2.13] are termed material tensors since only a dependence upon the particle size and shape is possible. In our case these tensors must reflect the geometry of the system (sphere and plane wall with unit vector $\mathbf{n}$). Expressed in an arbitrary cartesian coordinate system we consequently must have

$${}^tK_{ij} = K_{\|}n_in_j + K_{\perp}(\delta_{ij} - n_in_j), \tag{2.15a}$$

$${}^tR_{ij} = R\epsilon_{ij\mu}n_{\mu}, \tag{2.15b}$$

$${}^tQ_{ijk} = [Q_1\delta_{i\mu}n_{\nu} + Q_2n_in_{\mu}n_{\nu}]\Delta^{(2)}_{\mu\nu,jk}, \tag{2.15c}$$

$${}^rR_{ij} = R_{\|}n_in_j + R_{\perp}(\delta_{ij} - n_in_j), \tag{2.15d}$$

$${}^rQ_{ijk} = Q\epsilon_{i\mu\nu}n_{\nu}n_{\eta}\Delta^{(2)}_{\mu\eta,jk}, \tag{2.15e}$$

$$D_{ijkl} = D_1\Delta^{(2)}_{ij,kl} + D_2\Delta^{(2)}_{ij,\mu\nu}\Delta^{(2)}_{\nu\eta,kl}n_{\mu}n_{\eta} + D_3\Delta^{(2)}_{ij,\mu\nu}\Delta^{(2)}_{\eta\rho,kl}n_{\mu}n_{\nu}n_{\eta}n_{\rho}. \tag{2.15f}$$

A summation over repeated Greek indices is implied. Note that these tensors automatically satify [2.14], although this relation has not been used in deriving [2.14].

Eleven scalar coefficients are thus needed. Obviously they depend upon the distance d of

†The fact that only 6 independent tensors appear in [2.13] as well as the validity of [2.14] can be shown in an analogous fashion as for the case in which no wall is present (Hinch 1972).

the sphere center to the wall. For example, if the sphere is sufficiently far from the wall (such that the method of reflections can be used) one knows† (Lichtenthäler 1979)

$$K_{\|} = 6\pi a\left[1-\frac{9}{8}\left(\frac{a}{d}\right)+\frac{1}{2}\left(\frac{a}{d}\right)^3\right]^{-1},$$

$$K_{\perp} = 6\pi a\left[1-\frac{9}{16}\left(\frac{a}{d}\right)+\frac{1}{8}\left(\frac{a}{d}\right)^3\right]^{-1},$$

$$R = -\frac{3}{4}\pi a^2\left(\frac{a}{d}\right)^4,$$

$$Q_1 = -\frac{15}{4}\pi a^2\left(\frac{a}{d}\right)^2,$$

$$Q_2 = -\frac{15}{8}\pi a^2\left(\frac{a}{d}\right)^2,$$

$$R_{\|} = 8\pi a^3\left[1-\frac{1}{8}\left(\frac{a}{d}\right)^3\right]^{-1},$$

$$R_{\perp} = 8\pi a^3\left[1-\frac{5}{16}\left(\frac{a}{d}\right)^3\right]^{-1},$$

$$Q = \frac{5}{2}\pi a^3\left(\frac{a}{d}\right)^3,$$

$$D_1 = \frac{20}{3}\pi a^3\left[1-\frac{5}{16}\left(\frac{a}{d}\right)^3+\frac{1}{4}\left(\frac{a}{d}\right)^5\right]^{-1},$$

$$D_2 = \frac{20}{3}\pi a^3\left[\frac{5}{4}\left(\frac{a}{d}\right)^3+0\left(\left(\frac{a}{d}\right)^5\right)\right],$$

$$D_3 = \frac{20}{3}\pi a^3\left[\frac{5}{32}\left(\frac{a}{d}\right)^3+0\left(\left(\frac{a}{d}\right)^5\right)\right]. \qquad [2.16]$$

For some of the coefficients (like $K_{\perp}$, R and $R_{|}$) exact results are known. In all these cases it is found that [2.16] approximates these results quite well even for distances as low as one sphere diameter (Goldman *et al.* 1967). We thus expect qualitatively the same behavior for all the other coefficients, too.

Of special interest to us will be the stresslet for a freely suspended particle. Putting $\mathbf{F} = 0$ and $\mathbf{G} = 0$ one can readily invert [2.13a] and [2.13b] to obtain $\mathbf{u}_0$ and $\boldsymbol{\omega}$, respectively. Inserting this into [2.13c] we get by recalling [2.15]

$$\mathbf{s} = \mu_s\{D_1\mathbf{e} + \mathbf{\Delta}^{(2)}:[D_2'\mathbf{nn}\cdot\mathbf{e} + D_3'\mathbf{nnnn}:\mathbf{e}]\}, \qquad [2.17]$$

with

$$D_2' = D_2 + \frac{Q_1{}^2R_{\perp} + 2RQQ_1 + Q^2K_{\perp}}{R^2 - K_{\perp}R_{\perp}}, \qquad [2.18a]$$

and

$$D_3' = D_3 - \frac{(Q_1+Q_2)^2}{K_{\|}} - (D_2' - D_2). \qquad [2.18b]$$

†Accuracy up to the terms retained.

The difference of D_2' and D_3' to D_2 and D_3, respectively, is a reflection of the fact that the particle in general does not move with the fluid. The exception being a flow for which $\mathbf{n} \cdot \mathbf{e} = 0$. In this case the particle not only is carried along with the fluid, but the relation of $\mathbf{s}$ to $\mathbf{e}$ is then of the same form, as in the wall-free case (only the D_1 appears). This was the situation studied by Guth–Simha (1936).

3. THE STATISTICAL PROBLEM

Let us consider a two-dimensional shear flow of a highly dilute suspension of identical spheres of radius a. The flow shall be bounded by the planes $y = 0$ and $y = L$, respectively, where the upper plate is moving with constant velocity $V_w\mathbf{i}$. If $n(\mathbf{r})$ denotes the number of spheres (actually sphere centers) per unit volume at $\mathbf{r}$ the conservation equation

$$\frac{\partial}{\partial \mathbf{r}} \cdot (\mathbf{u}_0 n) = 0 \qquad [3.1]$$

applies in the steady state. The velocity $\mathbf{u}_0$ is determined by [2.13a, b] via a force and couple balance,

$$\mathbf{F} = -\mathbf{F}^{(\mathrm{Br})} = kT \frac{\partial}{\partial \mathbf{r}} \ln n,$$

$$\mathbf{G} = 0, \qquad [3.2]$$

where $\mathbf{F}^{(\mathrm{Br})}$ denotes the Brownian motion force. Consequently, since the unit normal of the wall is $\mathbf{j}$ for $y = 0$ and $-\mathbf{j}$ for $y = L$ it has to be of the form†

$$\mathbf{u}_0 = \mathbf{u}^0 + \alpha_1 \mathbf{j} \cdot \mathbf{e} + \alpha_2 \mathbf{jjj} : \mathbf{e} + \alpha_3 \mathbf{jj} \cdot \mathbf{F}^{(\mathrm{Br})} + \alpha_4 (\boldsymbol{\delta} - \mathbf{jj}) \cdot \mathbf{F}^{(\mathrm{Br})}. \qquad [3.3]$$

It is not necessary to list the α_i explicitely, but it is important to keep in mind that they are functions only of y. Since n itself can also depend only upon y, [3.3] reduces in our case to

$$\mathbf{u}_0 = \mathbf{i}\left[qy + \frac{1}{2}\alpha_1 q\right] + \mathbf{j}kT\alpha_3 \frac{\partial}{\partial y} \ln n, \qquad [3.4]$$

where $q = (V_w/L)$ is the applied shear rate. The diffusion equation [3.1] thus reads

$$\frac{\partial}{\partial y}\left(\alpha_3 \frac{\partial}{\partial y} n\right) = 0. \qquad [3.5]$$

Since the normal flux of particles to the walls has to be zero, i.e.

$$\alpha_3 \frac{\partial}{\partial y} n = 0, \text{ at } y = a \quad \text{and} \quad \text{at } y = L - a \qquad [3.6]$$

the solution to [3.5] is

$$n(\mathbf{r}) = \begin{cases} n_0 = \text{const.}, & a < y < L - a \\ 0, & \text{otherwise.} \end{cases} \qquad [3.7]$$

†Note that for the flow between 2 plates [2.15] still apply.

Thus, for the flow between two rigid parallel planes in steady relative shearing motion the distribution of spheres of a highly dilute suspension is uniform over the region accessible to the particles.

This result implies that no Brownian motion force is exerted upon the particles. Consequently equation [2.17] for the stresslet holds and simplifies for 2-dim. shear to

$$\mathbf{s} = \mu_s \frac{q}{2}\left(D_1 + \frac{1}{2}D_2'\right)(\mathbf{ij} + \mathbf{ji}) \tag{3.8}$$

for the stresslet holds.

4. THE SLIP VELOCITY FOR A SUSPENSION OF SPHERES

Let us again consider the situation described in section 3, namely the flow of a highly dilute suspension of spheres between two planes, say $y = 0$ and $y = L$, where the lower plane is at rest and the upper plane is moving with constant velocity $V_w\mathbf{i}$. If the spheres are much smaller than the distance between the planes, i.e.

$$0 < \frac{a}{L} \ll 1 \tag{4.1}$$

it makes sense to regard the suspension as a macro-continuum.

Using capital letters to characterize bulk quantities the bulk velocity $\mathbf{V}$ will obviously be of the form

$$\mathbf{V} = \mathbf{i}V(y) \tag{4.2}$$

satisfying the boundary conditions

$$V = 0 \qquad \text{at } y = 0, \tag{4.3a}$$

$$V = \frac{1}{2}V_w \qquad \text{at } y = \frac{L}{2}. \tag{4.3b}$$

While [4.3b] holds quite generally due to the symmetry of the problem (in the limit of [4.1]) [4.3a] assumes the the bulk fluid adheres to a solid surface, which is the customary assumption. By means of a macroscopic force balance the bulk shear stress has to be constant,

$$T_{yx} = \text{const.} \tag{4.4}$$

so that the problem is to relate T_{yx} and V.

It is clear that the Einsteinian expression (emphasized by the superscript E)

$$T_{yx}^{(\mathrm{E})} = \mu_s \frac{\mathrm{d}V}{\mathrm{d}y} + \frac{5}{2}\varphi_0\mu_s \frac{\mathrm{d}V}{\mathrm{d}y}, \tag{4.5a}$$

with

$$\varphi_0 = n_0 V_0 \tag{4.5b}$$

is justified only in the limit $(a/L) \to 0$ (φ_0 fixed), since in that case one is always far away from the wall. Since this hypothetical limit can never be achieved in practice let us try to see what happens if (a/l) is small, but finite. In this case, we need to consider only the half space $0 < y < (L/2)$ (see [4.3b]).

Clearly, far away from the wall T_{yx} has to approach its asymptotic value $T_{yx}^{(\mathrm{E})}$, i.e.

$$T_{yx} = T_{yx}^{(\mathrm{E})} \qquad \text{for } \frac{y}{a} \gg 1. \tag{4.6}$$

Closer to the wall, where the influence of the wall needs to be taken into consideration we expect area averages to represent properly the relevant bulk quantities. To this end we assume that A is a macrodifferential area parallel to the $y = 0$ plane. From a microscopic point of view its linear dimensions should be large in comparison to the average interparticle spacing. (It therefore cuts through both ambient fluid and particles.) Consequently the bulk rate of strain dyadic, E_{yx}, has to be defined as

$$\mathrm{E}_{yx} = \frac{1}{A}\left\{\int_{A-\Sigma A_p} \mathrm{d}A\, e_{yx}^{f} + \sum_{p} \int_{A_p} \mathrm{d}A\, e_{yx}^{p}\right\}. \tag{4.7}$$

The superscripts f and p, respectively, characterize the fluid and particle phase, A_p denotes that portion of A which cuts through a particle and the summation is over all particles which are intersected by A. Since $e_{yx}^{(p)} = 0$ for any rigid particle we can drop the second term. Similarly, since

$$t_{yx}^{f} = 2\mu_s e_{yx}^{f} \tag{4.8}$$

is the shear stress in the fluid phase, the definition of bulk shear stress reads

$$T_{yx} = \frac{1}{A}\left\{2\mu_s \int_{A-\Sigma A_p} \mathrm{d}A\, e_{yx}^{f} + \sum_{p} \int_{A_p} \mathrm{d}A\, t_{yx}^{p}\right\}. \tag{4.9}$$

Combining [4.7] and [4.9] and using the identity

$$\mathrm{E}_{yx} = \frac{1}{2}\frac{\mathrm{d}V}{\mathrm{d}y}, \tag{4.10}$$

we get

$$T_{yx} = \mu_s \frac{\mathrm{d}V}{\mathrm{d}y} + T_{yx}^{*} \tag{4.11}$$

with the particle contribution

$$T_{yx}^{*} = \frac{1}{A}\sum_{p} \int_{A_p} \mathrm{d}A[t_{yx}^{p}]. \tag{4.12}$$

Consequently, it is T_{yx}^{*} which poses problems. We know, however, the far away limit of T_{yx}^{*}, termed $T_{yx}^{*(\infty)}$ since by [4.5a] and [4.6] we have

$$T_{yx}^{*(\infty)} = \frac{5}{2}\varphi_0\mu_s \frac{\mathrm{d}V}{\mathrm{d}y}. \tag{4.13}$$

To demonstrate that this is indeed so we recall section 3. There it was shown that the (identical) spheres are unifromly distributed over the region accessible to them. This implies the relation

$$n = n(y) = n_0 H(y - a), \tag{4.14}$$

where $H(x)$ is the Heaviside step function.

If y denotes the distance of the averaging area to the wall [4.12] can thus be written as

$$T^*_{yx} = T^*_{yx}(y) = n_0 \int_{y-a}^{y+a} \mathrm{d}y^1 H(y^1 - a) \int_{A_1} \mathrm{d}A(t^p_{yx}), \tag{4.15}$$

with A_1 the circular area

$$x^2 + z^2 = a^2 - (y - y^1)^2. \tag{4.16}$$

Especially, if the averaging area A is more than one sphere diameter away from the wall the Heaviside function in [4.15] is identically equal to one (corresponding to the fact that the probability of A being intersected by a sphere whose center lies above it equals the probability of intersection by a sphere whose center lies below it). The integral of [4.15] thus becomes an integral over the volume of one sphere (recall [4.16]). By [2.12] the relation

$$T^*_{yx} = n_0 s_{yx}, \qquad \text{for} \quad y > 2a \tag{4.17}$$

to the stresslet follows (Batchelor 1970). Since s_{yx} is given by [3.8], the identity [4.13] emerges by using the far away limit of [2.16].†

Close to the wall, i.e. for $y < 2a$, [4.15] has to be used as it stands. Since it involves knowledge of shear stresses inside the particle even a simple problem like 2-dim. shear of a highly dilute suspension of spheres is in reality an extremely complicated one. Note that since T^*_{yx} depends upon y the shear rate must depend upon position, too.

The primary interest, however, of such a flow will clearly be the shear stress. Since this is constant we can obviously use its far away limit, i.e. Einstein's result. Furthermore, in this region the bulk shear rate has to be constant (otherwise T_{yx} will not be constant), say $q^{(\infty)}$,

$$\frac{\mathrm{d}V}{\mathrm{d}y} = q^{(\infty)} \qquad \text{for} \quad y \gg a. \tag{4.18}$$

Consequently we need to know the relation between the applied shear rate, (V_w/L), and $q^{(\infty)}$. However, by recalling the results obtained so far

$$T_{yx} = \mu_s \frac{\mathrm{d}V}{\mathrm{d}y} + T^*_{yx} = \mu_s q^{(\infty)} + T^{*(\infty)}_{yx} = \text{const.}, \tag{4.19}$$

a simple integration furnishes by means of [4.3] the desired relation in the form

$$q^{(\infty)} \frac{L}{2} + u_s = \frac{V_w}{2}, \tag{4.20}$$

†Since [4.15] already contains n_0 we can, in the expression [3.8] for s_{yx} replace the applied shear rate q by the bulk shear rate, since the difference between them is due to the particles.

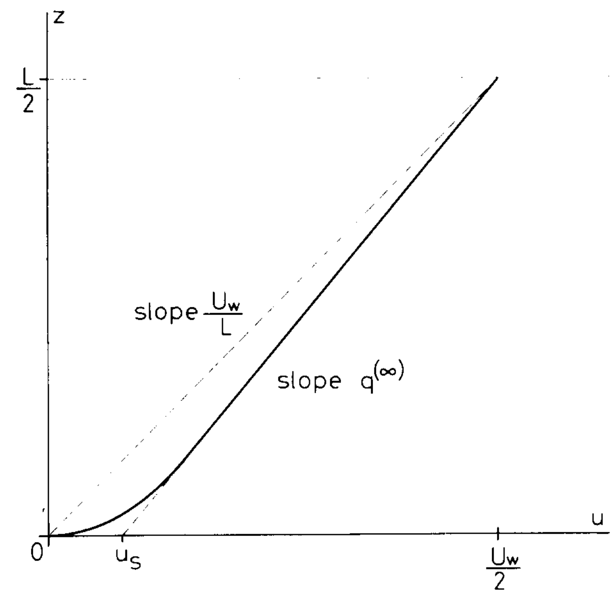

Figure 1.

with u_s the slip velocity† (see figure 1)

$$u_s = -\frac{1}{\mu_s} \lim_{(L/2)\to\infty} \int_0^{L/2} \mathrm{d}y (T^*_{yx} - T^{*(\infty)}_{yx}). \qquad [4.21]$$

Thus, to describe the flow properties of a suspension in the interior region and also to obtain correct average quantities in a bounded system one can indeed use classical continuum mechanical constitutive equations (which are always derived for an unbounded domain), provided a slip velocity is introduced. This has nothing to do with slipping of the true suspension, since [4.21] was derived via the assumption that the solution adheres to the wall. The slip velocity is simply a reflection of the fact that one has a bounded system. Failure to take this boundedness into account in the constitutive equations is accounted for via the introduction of the slip velocity, provided one is not interested in the detailed nature of the flow close to the wall. Should this be the case one does have to use constitutive equations taking the wall directly into account. This was the case studied by Tözeren and Skalak (1977).
useless.

Returning to our situation we insert [4.15] into [4.21]. Interchanging the order of the y and y' integration,

$$\int_0^\infty \mathrm{d}y \int_{y-a}^{y+a} \mathrm{d}y' H(y'-a) \ldots = \int_a^\infty \mathrm{d}y' \int_{y'-a}^{y'+a} \mathrm{d}y \ldots,$$

we again encounter an integral of t^p_{yx} over the volume of a sphere. By means of [2.12] and [4.12] this implies the result

$$u_s = \frac{5}{2} \varphi_0 q^{(\infty)} a - \frac{1}{\mu_s} \int_a^\infty \mathrm{d}y' \left[n_0 s_{yx}(y') - \frac{5}{2} \mu_s \varphi_0 q^{(\infty)} \right]. \qquad [4.22]$$

This expression, which has rigorously been derived, contains only the stresslet. It is a striking result because the region $y < 2a$ is the principal region in which wall effects occur. And within this region

†The limiting process $(L/2)\to\infty$ of [4.21] characterizes the situation (a/L) small but non-zero, we are considering.

[4.15] has to be used as it stands and cannot be cast in the form

$$T^*_{yx} = ns_{yx}, \qquad [4.23]$$

with n given by [4.14]. It is this reason which makes the evaluation of local quantities like T^*_{yx} so complicated (see Tözeren *et al.*, 1977). In our case we focus attention not on T^*_{yx} but rather on an average of that quantity (see [4.21]). This, and the interchange of the order of integration leads to [4.22]. Quite remarkably this equation also emerges if one uses the incorrect expression [4.23].

By means of [3.8] (recall also the footnote on p. 229 we can extract the parameter dependencies and the final expression for u_s becomes

$$u_s = \lambda\varphi_0 q^{(\infty)} a = \lambda\varphi_0 V_w \frac{a}{L} + 0(\varphi_0^2), \qquad [4.24a]$$

with the dimensionless slip coefficient

$$\lambda = \frac{5}{2}\left\{1 - \frac{1}{a}\int_a^\infty \mathrm{d}y' \left[\frac{D_1 + (1/2)D_2'}{5V_0} - 1\right]\right\}. \qquad [4.24b]$$

The slip velocity directly determines the apparent viscosity μ_a. This is defined by the relation of the shear stress to the applied shear rate (V_w/L), i.e.

$$T_{yx} = \mu_a \frac{V_w}{L}.$$

By means of [4.19] and [4.20] we get

$$\mu_a = \mu\left[1 - \frac{2u_s}{V_w}\right], \qquad [4.25]$$

with

$$\mu = \mu_s\left(1 + \frac{5}{2}\varphi_0\right), \qquad [4.26]$$

the true viscosity of the suspension. Consequently a positive slip coefficient λ (positive u_s) implies a decrease of μ_a by decreasing the dimensions of the viscosimeter. This effect, termed sigma phenomenon, is indeed observed for many suspensions (Goldsmith & Mason 1967). As far as a dilute suspension of spheres is concerned it is solely due to the fact that the behavior of a particle is directly influenced by the wall (see [2.12] and [2.14]). The other expected source of its origin, namely a totally different expression of the particle stress tensor T^*_{yx} close to the wall than far away from it apparently is immaterial as the remarks following [4.22] reveal. Although a particle free layer adjacent to the wall would also lead to the sigma effect such an "explanation" is not possible for a dilute suspension since by section 3 the spheres are uniformly distributed over the accessible region $y \geq a$.†

The expression [4.24b] for the slip velocity can also be written as

$$u_s = \frac{1}{\beta} T_{yx}, \qquad [4.27a]$$

with

$$\beta = \frac{u}{\lambda a \varphi_0}. \qquad [4.27b]$$

†Note, however, that the local bulk concentration (defined as an area average) is *not* constant for $y < 2a$.

This implies that slipping is resisted by a tangential force. Equation [4.27a], which on an *ad hoc* basis has been introduced by Lamb (1932), thus follows rigorously from our analysis. Generalizing it according to

$$\mathbf{u}_s - \mathbf{V}_w = \frac{1}{\beta}\mathbf{n}\times(\mathbf{n}\cdot\mathbf{T}\times\mathbf{n}), \qquad [4.28]$$

with $\mathbf{T}$ the bulk stress tensor of the disperse system ($T^{(E)}_{yx}$ in our case) and $\mathbf{V}_w$ the velocity of the boundary enables one to estimate the effect of small, but finite (a/L)-ratios for a variety of different flows (e.g. Brunn 1975).

This same generalization is possible if we introduce the concept of a particle depleted layer close to the wall. Although we prefer the slip velocity concept, since this comes out of the calculations unambiguously we could also insist upon retaining the no slip condition even if we use the Einsteinian result. In order to obtain correct average global qualities, however (nothing more can be expected) we then have to introduce (rather artificially) a layer of thickness D close to the wall, in which the viscosity is lower (for positive λ) than far away from the wall. For example we are free to attribute to that layer the solvent viscosity, which amounts to a particle free zone close to the wall. This being the case we get for $\varphi_0 \ll 1$

$$\lambda = \frac{5}{2}\frac{D}{a}, \qquad [4.29]$$

so that λ is essentially the ratio of this (fictitious) particle free zone to the sphere radius.

It is this thickness D for Vand (1948) lists the result

$$\frac{D}{a} = 1.301. \qquad [4.30]$$

Vand's calculation, however, rests on a number of assumptions, the basic three being:

(i) The integral appearing in [4.24b] can be evaluated by using [2.16], i.e. results obtained by the method of reflections. With the exception of a region immediately adjacent to the wall (of the order of one sphere diameter thick), the method of reflections applies. Thus, if $D_1 + (1/2)D_2'$ remains finite for $y \to a$ Vand's first assumption seems indeed reasonable, at least in order to obtain a first estimate of λ.

(ii) The particle moves with the fluid. This assumption certainly is not valid. It implies that the integrand of [4.24b] should be $D_1 + (1/2)D_2$ (see the remarks following [2.18]). Using (2.16) we see that the difference of $D_1 + (1/2)D_2$ to the correct expression of $D_1 + (1/2)D_2'$ is of the order $(a/y)^4$. Only by retaining nothing but the dominant, i.e. $(a/y)^3$-terms would this difference be immaterial (Vand, however, kept terms up to the order $(a/y)^5$).

(iii) A mirror-image suffices to obtain via the method of reflections the function $D_1 + (1/2)D_2'$. Vand himself realized that there was no rationale for this assumption since only the normal component of [2.3] can be satisfied with a mirror image. His hope that this would not matter very much, is unsubstantiated. Already at the dominant, i.e. $(a/y)^3$ power does $D_1 + (1/2)D_2'$ differ from the expression—$1 + (5/16)(a/y)^3$—listed by Vand (his [5.10]).

As a matter of fact, up to $0((a/y)^4)$† we deduce from [2.15]

$$\frac{D_1 + \frac{1}{2}D_2'}{5V_0} - 1 = \frac{15}{16}\left[\left(\frac{a}{y}\right)^3 - \frac{3}{16}\left(\frac{a}{y}\right)^4\right] + 0\left(\left(\frac{a}{y}\right)^5\right). \qquad [4.31]$$

†For reasons of consistency we have to stop at that order, although certain terms appearing in $D_1 + (1/2)D_2'$ can be guaranteed to a higher order.

If λ is estimated by using only the dominant term of [4.31], i.e. the $(a/y)^3$ term, then

$$\lambda = \frac{85}{64} = 1.33 \tag{4.32a}$$

including the $(a/y)^4$ term as well modifies this result to

$$\lambda = \frac{755}{512} = 1.47. \tag{4.32b}$$

Considering this to be a minor modification of [4.32a] suggests that the method of reflections suffices, at least as far as an estimate of the slip coefficient λ is concerned. Phrased differently, we do not expect a drastic change of our 1.4 approximation for λ, even though the wall layer $a < y < 2a$ is not at all properly represented by [4.31]. This conjecture is easily substantiated by using the detailed results of Tözeren *et al.* (1977). Extrapolating their numerical results for the far away bulk velocity up to the wall furnishes a value of 1.45 for λ.

Before ending let us look at the average viscosity $\langle\mu\rangle$, since this will enable us to see the fundamental mistake of Guth–Simha (1936) most clearly.

If we recall [4.11] and [4.17] we can, for $y > 2a$ define a local viscosity. By means of [3.8] this can be written as

$$\mu = \mu(y) = \mu_s\left\{1 + \frac{5}{2}nV_0\left(\frac{D_1 + \frac{1}{2}D_2'}{5V_0}\right)\right\}. \tag{4.33}$$

For $y < 2a$ no such relation can be written down. However, since [4.22] would have emerged from [4.21] even if the expression [4.23] had been employed it is tempting to define an average viscosity $\langle\mu\rangle$ by means of [4.33]. Thus, putting for $(a/L) \ll 1$

$$\langle\mu\rangle = \frac{2}{L}\int_0^{L/2} \mathrm{d}y\mu(y)$$

with $\mu(y)$ given by [4.33] we have

$$\langle\mu\rangle = \mu_a. \tag{4.34}$$

This is remarkable, for it tells us that the average viscosity (which by itself has no physical meaning) equals the apparent viscosity, which is the quantity observed in practice. It is not believed that this is a generally valid result, i.e. valid beyond the situation of a highly dilute suspension of spheres in 2-dim. shear.

The results of Guth–Simha (1936) seem to prove that point immediately. They studied a 2-dimensional dilational motion ($\mathbf{e}\cdot\mathbf{j} = 0$) of a highly dilute suspension of spheres.† Working entirely with $\mu(y)$ they list a result for the average viscosity, which, if equated with the apparent viscosity, corresponds to a negative slip coefficient. We found, however, a positive slip coefficient [4.32].

A close look at their calculations reveals the source of this discrepancy: their averaging procedure is wrong. This is readily demonstrated if we apply their way of averaging to our 2-dim. shear flow. Naively assuming that $\mu(y)$, defined by [4.33] can be used over the entire range and recalling [4.14] Guth–Simha (1936) postulate that the quantity μ_G, defined for $(a/L) \ll 1$ as

$$\mu_G = \mu_s\left\{\frac{2}{L}\int_0^{L/2}\mathrm{d}y + \frac{5}{2}\varphi_0\frac{1}{\frac{L}{2}-a}\int_a^{L/2}\mathrm{d}y\ \frac{D_1 + \frac{1}{2}D_2'}{5V_0}\right\},$$

†Note that for such a motion the rate of strain dyadic of the undisturbed motion has to depend upon position. Otherwise the adherence requirement at the wall cannot be met.

is the one observed in practice. But by using [4.24] we get for μ_G

$$\mu_G = \mu\left[1-\frac{2a}{L}\varphi_0\lambda_G\right], \tag{4.35}$$

with $\lambda_G = \lambda - (5/2)$.

Thus, since μ_G has nothing to do with the observed quantity μ_a (involving λ instead of λ_G) it is immaterial that λ_G is negative (-1.05, according to our estimate).

SUMMARY AND DISCUSSION

For a dilute suspension of spheres the theory described in the previous sections takes particle-wall interactions explicitly into account. The resulting concept of a slip velocity u_s for the suspension is not new (e.g. Cox *et al.* 1971). What is new is the fact that this slip velocity can be obtained without detailed knowledge of the bulk velocity. Although one can use that approach, too (Tözeren *et al.* 1977), it is far more economical to focus attention on u_s from the beginning.

As our main result, namely [4.22] and [4.24], respectively, shows, only the stresslet **S** needs to be known. This is striking, for in the immediate vicinity of the wall ($a < d < 2a$) the particle contribution to the bulk stress–tensor cannot be related to the stresslet. Yet, if one ignores that fact the correct expression for u_s still emerges.

The slip velocity u_s can be cast in a form first suggested by Lamb, 1932. It is found that Lamb's coefficient of sliding friction is ([4.27b])

$$\frac{u}{1.45a\Phi_0}$$

$$\frac{\mu}{1.45a\varphi_0}$$

where μ is the viscosity of the suspension, a the radius of a sphere and φ_0 the volume fraction of spheres ($\varphi_0 \ll 1$). Being positive, the apparent viscosity of a spherical suspension will thus decrease with decreasing dimensions of any viscometer. This is the so-called sigma phenomenon.

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